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On resolving the multiplicity of tensor products of irreducible representations of symplectic groups

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Abstract. The multiplicity problem of r-fold tensor products of irreducible representations of $\mathrm{Sp}(2k, \boldsymbol{C})$ are considered. The arbitrary r-fold tensor product is shown to be isomorphic to a subspace of the holomorphic Hilbert (Bargmann) space of $n \times 2k$ complex variables. Maps are constructed which carry an irreducible representation of $\mathrm{Sp}(2k, \boldsymbol{C})$ into this subspace. An algebra of commuting operators is constructed and it is shown how eigenvalues and eigenvectors of certain of these operators can be used to resolve the multiplicity.

1. Introduction

One of the outstanding problems in the representation theory of Lie groups is the multiplicity problem. In decomposing tensor products of irreducible representations of a group, the same irreducible representation may appear more than once; the problem is to find a canonical way of treating the equivalent representations which may occur in this decomposition. Several authors have analysed this problem for the unitary groups from several points of view [1–4]. Recently, Klink and Ton-That [5] have given a method to decompose tensor products of irreducible representations of $\mathrm{U}(N)$ by using a certain class of commuting differential operators. Underlying their method is the use of polynomial realizations of all irreducible representations of the $\mathrm{U}(N)$ groups. Such polynomial realization has the advantage of being basis independent. Moreover, the method can be implemented on a computer.

In this paper, we shall give a procedure for decomposing r-fold tensor products of irreducible representations of $\mathrm{Sp}(2k, \boldsymbol{C})$, the complex $2k \times 2k$ symplectic groups, by exhibiting a general class of commuting Casimir operators. The main tools needed to carry out this analysis are polynomial realizations of tensor products and the theory of dual pairs [6], which is used to construct the algebra of Casimir operators. Our work follows the spirit of [5].

The general setup of the problem will be discussed in section 2. Section 3 makes use of the notion of dual pairs to exhibit an algebra of commuting operators. Certain elements of this algebra are Hermitian; the eigenvalues and eigenvectors of these elements can then be used to resolve the multiplicity. The procedure is shown explicitly in section 4 with an example of representations of $\mathrm{Sp}(4, \boldsymbol{C})$.

2. The general setup

Let G denote the complex $2k \times 2k$ symplectic group. It is well known that an irreducible holomorphically induced representation of G is uniquely determined by a k-tuple of non-negative integers $(m)=(m_1,\ldots,m_k)$, called the signature of the representation, which satisfies the dominant condition $m_1\geqslant\ldots\geqslant m_k$ ([10]). In general, a concrete realization of such an irreducible representation of G can be obtained as follows. Suppose $(m)=(m_1,\ldots,m_k)$ is a signature of G such that $m_{q+1}=\ldots=m_k=0$ for some $1\leqslant q\leqslant k$. Let $C^{q\times 2k}$ be the vector space of all $q\times 2k$ complex matrices and $S(C^{q\times 2k*})$ be the symmetric algebra of all polynomial functions on $C^{q\times 2k}$. Let B_q denote the lower triangular subgroup of $\mathrm{GL}(q,C)$, the $q\times q$ general linear group. We define a holomorphic character

$$\xi^{(m)}: B_q\to C^*$$
$$\xi^{(m)}(b)=b_{11}^{m_1}\ldots b_{qq}^{m_q}\qquad b\in B_q$$

where b_{ii} is the ith diagonal element of b. Now, consider the following vector space:

$$V_{\mathrm{Sp}}^{(m)}=\left\{f\in S(C^{q\times 2k*})|f(bX)=\xi^{(m)}(b)f(X)\qquad \text{for } b\in B_q,\ X\in C^{q\times 2k}\right.$$
$$\left.\text{and } \sum_{p=1}^{k}\left(\frac{\partial^2 f}{\partial Z_{ip}\partial Z_{jp+k}}-\frac{\partial^2 f}{\partial Z_{ip+k}\partial Z_{jp}}\right)=0,\ 1\leqslant i<j\leqslant q\right\}.$$

Then according to [16], the representation $R_{\mathrm{Sp}}^{(m)}$ of G on $V_{\mathrm{Sp}}^{(m)}$ by right translation, that is, $(R_{\mathrm{Sp}}^{(m)}(g)f)(Z)=f(Zg)$, for all $(g,Z)\in \mathrm{G}\times C^{q\times 2k}$, is irreducible with signature (m). Hence, our representation space of G is a polynomial space.

Next, we consider r-fold tensor products of arbitrary irreducible representations of G with signatures $(M_{(1)}),\ldots,(M_{(r)})$, respectively, where each label $M_{(i)}$ is an k-tuple of integers $(M_{i_1},\ldots,M_{i_k})$, $1\leqslant i\leqslant r$. We discard those M_{ij}, $1\leqslant i\leqslant r$, $1\leqslant j\leqslant k$, which are equal to zero, and relabel the indices so that they form an n-tuple of integers of the form

$$(M)=(M_1,\ldots,M_{p_1},M_{p_1+1},\ldots,M_{p_1+p_2},\ldots,M_n)$$

where $M_1,\ldots,M_{p_1}$ are the p_1 non-zero elements of $M_{(1)}$, and $M_{p_1+1},\ldots,M_{p_1+p_2}$ are the p_2 non-zero elements of $M_{(2)}$ and so on such that $p_1+\ldots+p_r=n$.

Let $C^{n\times 2k}$ denote the vector space of all $n\times 2k$ complex matrices and $\mathscr{F}\equiv\mathscr{F}(C^{n\times 2k})$ denote the Fock space of $n\times 2k$ complex variables, as constructed in [5].

If D_n denotes the group of all complex diagonal invertible matrices of order n, and if $(M)=(M_1,\ldots,M_n)$ is the n-tuple of non-negative integers as before, we define a holomorphic character

$$\zeta^{(M)}:D_n\to C^*$$
$$\zeta^{(M)}(d)=d_{11}^{M_1}\ldots d_{nn}^{M_n}\qquad \forall d\in D_n.$$

A polynomial function $p:C^{n\times 2k}\to C$ is said to transform covariantly with respect to $\zeta^{(M)}$ if $f(\mathrm{d}Z)=\zeta^{(M)}(\mathrm{d})f(Z)$, for all (d,Z) belonging to $D_n\times C^{n\times 2k}$. The polynomial functions which transform covariantly with respect to $\zeta^{(M)}$ form a subspace of $\mathscr{F}$, denoted by $P^{(M)}\equiv P^{(M)}(C^{n\times 2k})$. Let $R_0^{(m)}$ denote the representation of G on $P^{(M)}$ by right translation. By an argument similar to the one used in the proof of theorem 2.4 in [7], we have the following:

Theorem 2.1. The G-module $P^{(M)}$ is isomorphic to the Kronecker tensor product $V_{\mathrm{Sp}}^{(M_1,0,\ldots,0)}\otimes\ldots\otimes V_{\mathrm{Sp}}^{(M_n,0,\ldots,0)}$.

Remark 2.2. As will be shown later, it is convenient to have an explicit formula which gives the direct sum decomposition of the tensor product $(m_1, \ldots, m_k)\otimes(n, 0, \ldots, 0)$ of G. There are many well-known formulas for the decomposition of such a tensor product [8, 9]. Using a result in [8], we have derived the following formula for the decomposition of tensor product $(m_1, \ldots, m_k)\otimes(n, 0, \ldots, 0)$, which we call the Weyl formula of symplectic groups (see [7] for details):

$$(m_1, m_2, \ldots, m_k)\otimes(n, 0, \ldots, 0)$$
$$\cong\sum\oplus(m_1+a_1-a_{2k}, m_2+a_2-a_{2k-1}, m_3+a_3-a_{2k-2}, \ldots, m_k+a_k-a_{k+1})$$

where the sum is over all integers a_i, $i=1, \ldots, 2k$, subject to the following conditions:

$$a_1+\ldots+a_{2k}=n$$
$$0\leqslant a_i\leqslant m_{i-1}-m_i-a_{2k-(i-2)}+a_{2k-(i-1)}$$
$$0\leqslant a_{2k-j}\leqslant m_{j+1}-m_{j+2}$$
$$0\leqslant a_{k+1}\leqslant m_k$$

where $i=2, 3, \ldots, k$ and $j=0, 1, \ldots, k-2$.

All the inequalities above are necessary for making sure that the signatures in the direct sum satisfy the dominant conditions. These inequalities arose when we derived our formula, which involved the consideration of certain Young tableaux. This formula is an analogue of the Weyl formula for $\mathrm{U}(N)$ in [10, section 80]. Now, if $m_1, \ldots, m_k$ and n are large integers, then the tensor product decomposition of $(m_1, \ldots, m_k)\otimes(n, 0, \ldots, 0)$ will involve many terms and the calculation of such a decomposition is a tedious process. The advantage of the above formula is that we can easily write a computer program to perform the calculation. If we use the above formula repeatedly n times, then we can easily obtain the direct sum decomposition of the G-module

$$P^{(M)}\cong V_{\mathrm{Sp}}^{(M_1,0,\ldots,0)}\otimes\ldots\otimes V_{\mathrm{Sp}}^{(M_n,0,\ldots,0)}.$$

Now, for $1\leqslant i\leqslant r$, let B_{p_i} denote the lower triangular (Borel) subgroup of $\mathrm{GL}(p_i, \boldsymbol{C})$. Let $H_0^{(M)}$ denote the subspace of $P^{(M)}$ which consists of all polynomial functions f which satisfy the covariant condition

$$f\left(\begin{bmatrix} b_{p_1} & & 0 \\ & \ddots & \\ 0 & & b_{p_r}\end{bmatrix} Z\right)=b_{11}^{M_1}\ldots b_{nn}^{M_n}f(Z) \tag{2.3}$$

where b_{jj}, $1\leqslant j\leqslant n$, denotes the jth diagonal entry of the matrix

$$\begin{bmatrix} b_{p_1} & & 0 \\ & \ddots & \\ 0 & & b_{p_r}\end{bmatrix}.$$

Also, let $(M_i)_{\mathrm{GL}}\equiv(M_{i_1}, \ldots, M_{i_k}, 0, \ldots, 0)$ be a $2k$-tuple of non-negative integers, for $1\leqslant i\leqslant r$. In [5], Klink and Ton-That proved the following:

Theorem 2.4. The $\mathrm{GL}(2k, \boldsymbol{C})$-module $(M_1)_{\mathrm{GL}}\otimes\ldots\otimes(M_r)_{\mathrm{GL}}$ is isomorphic to the $\mathrm{GL}(2k, \boldsymbol{C})$-module $(R_{\mathrm{GL}0}^{(M)}, H_0^{(M)})$ where $\mathrm{GL}(2k, \boldsymbol{C})$ acts on $H_0^{(M)}$ by right translation.

Now, consider the following vector subspace of $P^{(M)}$:

$$H^{(M)}(C^{n\times 2k})=\left\{f\in H_0^{(M)} \mid D_{\alpha_{p_i}\beta_{p_i}}f=\sum_{l=1}^{k}\left(\frac{\partial^2}{\partial Z_{\alpha_{p_i}l+k}\,\partial Z_{\beta_{p_i}l}}-\frac{\partial^2}{\partial Z_{\alpha_{p_i}l}\,\partial Z_{\beta_{p_i}l+k}}\right)f=0,\right.$$
$$\left.\text{for } \alpha_{p_i}<\beta_{p_i},\ \sum_{s=1}^{i-1}p_s+1\leqslant\alpha_{p_i},\ \beta_{p_i}\leqslant\sum_{s=1}^{i}p_s,\ 1\leqslant i\leqslant r\right\}.$$

To simplify notation, we let $H^{(M)}\equiv H^{(M)}(C^{n\times 2k})$. Then we have the following:

Theorem 2.5 The r-fold tensor product $V_{\mathrm{Sp}}^{(M_{(1)})}\otimes\ldots\otimes V_{\mathrm{Sp}}^{(M_{(r)})}$ is isomorphic to the G-module $H^{(M)}$, where G acts on $H^{(M)}$ by right translation.

Proof. Let (m) be a $2k$-tuple of non-negative integers that satisfies the dominant condition. Let q be the number of non-zero elements in (m). Consider the vector space

$$V_{\mathrm{GL}}^{(m)}=\{f: C^{q\times 2k}\to C \mid q\leqslant 2k, f \text{ is a polynomial function},$$
$$f(bX)=\xi^{(m)}(b)f(X),\ \forall(b,X)\in B_q\times C^{q\times 2k}\}.$$

Then according to [11], the representation $R_{\mathrm{GL}}^{(m)}$ of GL($2k$, C) on $V_{\mathrm{GL}}^{(m)}$ by right translation is irreducible with signature (m). Also, $V^{(m)}$ is a subspace of $V_{\mathrm{GL}}^{(m)}$, if $q\leqslant k$.

Now, define a map

$$\varphi\colon V_{\mathrm{GL}}^{(M_1)\mathrm{GL}}\otimes\ldots\otimes V_{\mathrm{GL}}^{(M_r)\mathrm{GL}}\to H_0^{(M)}$$

by

$$\varphi(f_1\otimes\ldots\otimes f_r)\begin{pmatrix}Z_1\\ \vdots\\ Z_r\end{pmatrix}=f_1(Z_1)\ldots f_r(Z_r)$$

for $Z_i\in C^{p_i\times 2k}$, $1\leqslant i\leqslant r$ and extend φ linearly. Now, we have

$$\varphi(f_1\otimes\ldots\otimes f_r)\begin{pmatrix}b_1 & & \\ & \ddots & Z\\ & & b_r\end{pmatrix}=f_1(b_1Z_1)\otimes\ldots\otimes f_r(b_rZ_r)$$
$$=b_{11}^{M_1}\ldots b_{p_1p_1}^{M_{p_1}}f_1(Z_1)b_{p_1+1p_1+1}^{M_{p_1+1}}\ldots f_2(Z_2)\ldots b_{nn}^{M_n}f_r(Z_r)$$
$$=b_{11}^{M_1}\ldots b_{nn}^{M_n}f_1(Z_1)\ldots f_r(Z_r)$$
$$=b_{11}^{M_1}\ldots b_{nn}^{M_n}\varphi(f_1\otimes\ldots\otimes f_r)(Z)$$

for $b_i\in B_{p_i}$, $Z_i\in C^{p_i\times 2k}$, $Z\in C^{n\times 2k}$ and $1\leqslant i\leqslant r$.

Hence, $\varphi(V_{\mathrm{GL}}^{(M_1)\mathrm{GL}}\otimes\ldots\otimes V_{\mathrm{GL}}^{(M_r)\mathrm{GL}})$ is in $H_0^{(M)}$. GL($2k$, C) acts on $V_{\mathrm{GL}}^{(M_1)\mathrm{GL}}\otimes\ldots\otimes V_{\mathrm{GL}}^{(M_1)\mathrm{GL}}$ by $R_{\mathrm{GL}}^{(M)\mathrm{GL}}(g)\ (f_1\otimes\ldots\otimes f_n)=R_{\mathrm{GL}}^{(M_1)\mathrm{GL}}(g)f_1\otimes\ldots\otimes R_{\mathrm{GL}}^{(M_r)\mathrm{GL}}f_n$.

Now, let $g\in\mathrm{GL}(k, C)$, then we have

$$R_{\mathrm{GL}0}^{(M)}(g)\varphi(f_1\otimes\ldots\otimes f_r)(Z)=\varphi(f_1\otimes\ldots\otimes f_r)(Zg)$$
$$=f_1(Z_1g)\ldots f_r(Z_rg)$$
$$=R_{\mathrm{GL}}^{(M_1)\mathrm{GL}}(g)f_1(Z_1)\ldots R_{\mathrm{GL}}^{(M_r)\mathrm{GL}}(g)f_r(Z_r)$$
$$=\varphi(R_{\mathrm{GL}}^{(M_1)\mathrm{GL}}(g)f_1\otimes\ldots\otimes R_{\mathrm{GL}}^{(M_r)\mathrm{GL}}(g)f_r)(Z)$$
$$=(\varphi(R_{\mathrm{GL}}^{(M)}(g)(f_1\otimes\ldots\otimes f_r)))(Z).$$

Therefore, φ is an intertwining map with respect to GL$(2k, C)$ and this implies φ is also an intertwining map with respect to G. Clearly, φ is one-to-one by definition. By theorem 2.4, $V_{\mathrm{GL}}^{(M_1)\mathrm{GL}}\otimes\ldots\otimes V_{\mathrm{GL}}^{(M_r)\mathrm{GL}}$ and $H_0^{(M)}$ have the same dimension, hence φ is a G-module isomorphism. Now, if $h_i\in V_{\mathrm{GL}}^{(M_i)\mathrm{GL}}$, then $\varphi(h_1\otimes\ldots\otimes h_r)=h_1(z_1)\ldots h_r(z_r)$ by the definition of the map φ. Suppose we have

$$D_{\alpha_{p_i}\beta_{p_i}}h_i=0 \qquad \text{for } h_i\in V_{\mathrm{Sp}}^{(M_{(i)})} \qquad \alpha_{p_i}<\beta_{p_i} \qquad 1\leqslant i\leqslant r.$$

Then

$$\begin{aligned} D_{\alpha_{p_i}\beta_{p_i}}\varphi(h_1\otimes\ldots\otimes h_r) &= D_{\alpha_{p_i}\beta_{p_i}}h_1\ldots h_r \\ &= h_1\ldots h_{i-1}h_{i+1}\ldots h_r D_{\alpha_{p_i}\beta_{p_i}}h_i \\ &= 0. \end{aligned}$$

Therefore, we have $\varphi(V_{\mathrm{Sp}}^{(M_{(1)})}\otimes\ldots\otimes V_{\mathrm{Sp}}^{(M_{(r)})})\subset H^{(M)}\subset H_0^{(M)}$. Since φ is a G-module isomorphism, every element in $H_0^{(M)}$ is completely determined by its pre-image. So, if $f\in H^{(M)}$, then the pre-image of f must be in $V_{\mathrm{Sp}}^{(M_{(1)})}\otimes\ldots\otimes V_{\mathrm{Sp}}^{(M_{(r)})}$. Hence, we have $\varphi(V_{\mathrm{Sp}}^{(M_{(1)})}\otimes\ldots\otimes V_{\mathrm{Sp}}^{(M_{(r)})})=H^{(M)}$ and the result follows. Q.E.D.

As we shall see later, theorem 2.1, remark 2.2 and theorem 2.5 are very important in the decomposition of the tensor product $V_{\mathrm{Sp}}^{(M_{(1)})}\otimes\ldots\otimes V_{\mathrm{Sp}}^{(M_{(r)})}$. Since the space $H^{(M)}$ is a subspace of $P^{(M)}$, theorem 2.1 gives an upper bound for the multiplicity of an irreducible representation of G occuring in the tensor product. We conclude this section with the following:

Definition 2.6. Let $V_{\mathrm{Sp}}^{(m)}$ be a G-module. The isotypic component $I(V_{\mathrm{Sp}}^{(m)})$ of $V_{\mathrm{Sp}}^{(m)}$ in $\mathscr{F}$ is the sum of all G-modules in $\mathscr{F}$ which are equivalent to $V_{\mathrm{Sp}}^{(m)}$.

3. The multiplicity breaking of the tensor product

We shall now give a procedure for explicit decomposition of the r-fold tensor product (or equivalently the G-module $H^{(M)}$), by using the theory of dual pairs. First, we let $L^{(M)}$ denote the representation of GL(n, C) on $P^{(M)}$ defined by $(L^{(M)}(g)p)(Z)=p(g^{-1}Z)$, $g\in\mathrm{GL}(n, C)$ and $R^{(M)}$ denote the representation of G on $P^{(M)}$ by right translation. If we let L_{ij} (respectively, R_{rs}) denote the infinitesimal operators of $L^{(M)}$ (respectively, $R^{(M)}$) corresponding to the standard basis e_{ji} (respectively, e_{rs}) of the Lie algebra $C^{n\times n}$ (respectively, $C^{2k\times 2k}$) of GL(n, C) (respectively, GL$(2k, C)$); then we have

$$L_{ij}=\sum_{\eta=1}^{2k} Z_{i\eta}\frac{\partial}{\partial Z_{j\eta}}$$

$$R_{rs}=\sum_{\eta=1}^{n} Z_{\eta r}\frac{\partial}{\partial Z_{\eta s}} \qquad 1\leqslant i \qquad j\leqslant n \qquad 1\leqslant r \qquad s\leqslant 2k.$$

Among these operators, we have the particular operators

$$L_{\alpha_{p_i}\beta_{p_i}} \qquad \text{where } \sum_{k=1}^{i-1} p_k+1\leqslant\alpha_{p_i} \qquad \beta_{p_i}\leqslant\sum_{k=1}^{i} p_k$$

which correspond to the infinitesimal operators of the subgroups $\mathrm{GL}(p_i, C)$, $1 \leqslant i \leqslant r$, of $\mathrm{GL}(n, C)$. It was shown in [5] that $H_0^{(M)}$ consists of all polynomial functions in $P^{(M)}$ which are simultaneously annihilated by all operators of the form

$$L_{\alpha_{p_i}\beta_{p_i}} \qquad \text{with } \alpha_{p_i} < \beta_{p_i} \qquad 1 \leqslant i \leqslant r.$$

Therefore, the subspace $H^{(M)}$ of $H_0^{(M)}$ consists of all polynomial functions in $P^{(M)}$ which are simultaneously annihilated by all operators of the form

$$L_{\alpha_{p_i}\beta_{p_i}} \qquad \text{and } D_{\alpha_{p_i}\beta_{p_i}} \qquad \text{with } \alpha_{p_i} < \beta_{p_i} \qquad 1 \leqslant i \leqslant r. \tag{3.1}$$

Now, let $\mathrm{SU}(n, n)$ be the linear isometry group for the Hermitian form $|Z_1|^2 + |Z_2|^2 + \ldots + |Z_n|^2 - |Z_{n+1}|^2 - \ldots - |Z_{2n}|^2$ over C. Then the group $\mathrm{SO}^*(2n)$ is the set of all elements g in $\mathrm{SU}(n, n)$ such that $g^T J g = J$, where g^T is the transpose of g and

$$J = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}$$

where I_n is the $n \times n$ identity matrix. Then according to [6, 12], $(\mathrm{SO}^*(2n), \mathrm{Sp}(2k, C))$ forms a dual pair of reductive groups. Let

$$L_{ij} = \sum_{\eta=1}^{2k} Z_{i\eta} \frac{\partial}{\partial Z_{j\eta}} \qquad P_{ij} = \sum_{\eta=1}^{k} (Z_{i\eta+k} Z_{j\eta} - Z_{i\eta} Z_{j\eta+k})$$

and

$$D_{ij} = \sum_{\eta=1}^{k} \left(\frac{\partial^2}{\partial Z_{i\eta+k}\, \partial Z_{j\eta}} - \frac{\partial^2}{\partial Z_{i\eta}\, \partial Z_{j\eta+k}} \right) \qquad 1 \leqslant i \qquad j \leqslant n. \tag{3.2}$$

These operators form a basis for the Lie algebra $\mathrm{so}^*(2n)$ of the group $\mathrm{SO}^*(2n)$ and they generate a universal enveloping algebra $\mathscr{U}$ of differential operators which acts on $\mathscr{F}$. Moreover, by the Poincare–Birkoff–Witt theorem, the ordered monomials in L_{ij}, P_{ij}, D_{ij} form a basis for the algebra $\mathscr{U}$.

Remark 3.3. In [5], Klink and Ton-That have used the dual pair $(\mathrm{U}(n), \mathrm{GL}(N, C))$ for their decomposition of tensor products. Since $\mathrm{Sp}(2k, C)$ is a subgroup of $\mathrm{GL}(2k, C)$, the dual group $\mathrm{SO}^*(2n)$ contains $\mathrm{U}(n)$ as a subgroup. It is easy to see that the operators L_{ij} form a basis for the Lie algebra of $\mathrm{U}(n)$. As we shall see later, the addition of the P_{ij} and D_{ij} operators makes it possible for us to generalize the result in [5].

Now, assume the number of times an irreducible representation $R_{\mathrm{Sp}}^{(m)}$ of G occurs in $H^{(M)}$ is known (this is the Clebsch–Gordan series problem; there are a number of ways of obtaining a closed-form formula for this multiplicity [9, 13]). However, those close form formulae require the use of Young tableaux and it is, in general, difficult to compute when G is large. Moreover, it is not easy to perform such a calculation on a computer. In contrast, to compute the multiplicity of $R_{\mathrm{Sp}}^{(m)}$ in $P^{(M)}$ using theorem 2.1 is a straightforward procedure which involves only the formula stated in remark 2.2. As mentioned before, it is possible and natural to implement such a calculation on a computer. As we shall see later, the multiplicity of $R_{\mathrm{Sp}}^{(m)}$ in $P^{(M)}$ together with projection operators in (3.1) lead us immediately to the multiplicity of $R_{\mathrm{Sp}}^{(m)}$ in $H^{(M)}$.

Now, suppose the G-module $(R_{\mathrm{Sp}}^{(m)}, V_{\mathrm{Sp}}^{(m)})$ of G occurs in $P^{(M)}$ with multiplicity μ. Then from a consequence of Burnside's theorem and the theory of dual pairs, there exist μ linearly independent elements in $\mathscr{U}$ which form a basis for the vector space $\mathrm{Hom}_G(V_{\mathrm{Sp}}^{(m)}, P^{(M)})$ of all intertwining operators from $V_{\mathrm{Sp}}^{(m)}$ to $P^{(M)}$. To simplify our exposition, it suffices to consider the highest weight vector $h_{\max}^{(m)}$ of $V_{\mathrm{Sp}}^{(m)}$, then it follows that we can choose μ elements $p_1, \ldots, p_\mu$ of $\mathscr{U}$ such that $p_i h_{\max}^{(m)}$, $1 \leqslant i \leqslant \mu$, are linearly independent highest weight vectors of the μ copies of the G-module equivalent to $V_{\mathrm{Sp}}^{(m)}$ which are contained in $P^{(M)}$. In order to obtain an orthogonal direct sum decomposition of $I(V_{\mathrm{Sp}}^{(m)}) \cap H^{(M)}$, the intersection of the isotypic component of $V_{\mathrm{Sp}}^{(m)}$ with $H^{(M)}$, we must find Hermitian operators in $\mathscr{U}$ that leave $H^{(M)}$ invariant and that decompose 'canonically' $I(V_{\mathrm{Sp}}^{(m)}) \cap H^{(M)}$ into distinct eigenspaces. For this, we let $W_{\max}^{(m)(M)}$ denote the vector space spanned by $p_i h_{\max}^{(m)}$, and let $\mathrm{Ker}_{\max}^{(m)(M)}$ denote its projection in $H^{(M)}$, that is, $\mathrm{Ker}_{\max}^{(m)(M)}$ is the common kernel subspace in $W_{\max}^{(m)(M)}$ of all operators $L_{\alpha_{p_i}\beta_{p_i}}$ and $D_{\alpha_{p_i}\beta_{p_i}}$ with $\alpha_{p_i} < \beta_{p_i}$, $1 \leqslant i \leqslant r$.

We shall find operators in $\mathscr{U}$ which commute with the above differential operators and decompose $\mathrm{Ker}_{\max}^{(m)(M)}$ into distinct one-dimensional eigenspaces.

Remark 3.4. In [5], Klink and Ton-That decomposed the GL$(2k, C)$-module $H_0^{(M)}$ by finding operators that commute with $L_{\alpha_{p_i}\beta_{p_i}}$, $1 \leqslant i \leqslant r$. The main difference in our paper is that we now decompose the subspace $H^{(M)}$ of $H_0^{(M)}$. We need to find operators in $\mathscr{U}$ that commute with $D_{\alpha_{p_i}\beta_{p_i}}$ in addition to $L_{\alpha_{p_i}\beta_{p_i}}$.

The $D_{\alpha_{p_i}\beta_{p_i}}$ operators come into our consideration naturally since they are a part of the basis for the Lie algebra SO*$(2n)$. Let us first concentrate on the operators which commute with $L_{\alpha_{p_i}\beta_{p_i}}$, with $\alpha_{p_i} < \beta_{p_i}$, $1 \leqslant i \leqslant r$. We set

$$R_{\alpha\beta}^{(p_i)} = \sum_{\eta} z_{\eta\alpha} \frac{\partial}{\partial z_{\eta\beta}} \qquad 1 \leqslant \alpha \qquad \beta \leqslant 2k \tag{3.5}$$

where η ranges over all rows of the submatrix Z_i for a fixed index i, $1 \leqslant i \leqslant r$. Let $R^{(p_i)}$ denote the $2k \times 2k$ matrix

$$\begin{pmatrix} R_{11}^{(p_i)} & \cdots & R_{12k}^{(p_i)} \\ \vdots & R_{\alpha\beta}^{(p_i)} & \vdots \\ R_{2k1}^{(p_i)} & \cdots & R_{2k2k}^{(p_i)} \end{pmatrix}.$$

Consider the following $n \times n$ matrices:

$$P = [P_{\alpha\beta}] \qquad D = [D_{\alpha\beta}] \qquad L = [L_{\alpha\beta}] \qquad \text{and } E = [E_{\alpha\beta}] = [-L_{\beta\alpha}]$$

where $P_{\alpha\beta}$, $D_{\alpha\beta}$ and $L_{\alpha\beta}$ are defined in (3.2).

Now, let us partition P into block matrices of the form

$$P = \begin{pmatrix} [P]_{11} & \cdots & [P]_{1r} \\ \vdots & [P]_{uv} & \vdots \\ [P]_{r1} & \cdots & [P]_{rr} \end{pmatrix}$$

where each $[P]_{uv}$ is a $p_u \times p_v$ matrix, $1 \leqslant u$, $v \leqslant r$. Similarly, we partition D, E, L into block matrices exactly like P. Then, we have the following:

Theorem 3.6. In the universal enveloping algebra $\mathscr{U}$, consider the trace of the product of the matrices of the forms:

$$\begin{aligned}
&(1)\quad [L]_{u_1u_2}\ [L]_{u_2u_3}\ \ldots\ [L]_{u_iu_{i+1}}\ \ldots\ [L]_{u_qu_{q+1}}\\
&(2)\quad [P]_{v_1v_2}\ [D]_{v_2v_3}\ [P]_{v_3v_4}\ [D]_{v_4v_5}\\
&\qquad\qquad \ldots\ [P]_{v_sv_{s+1}}\ [D]_{v_{s+1}v_{s+2}}\ \ldots\ [P]_{v_qv_{q+1}}\ [D]_{v_{q+1}v_{q+2}}\\
&(3)\quad [P]_{t_1t_2}\ [E]_{t_2t_3}\ \ldots\ [E]_{t_jt_{j+1}}\ \ldots\ [E]_{t_qt_{q+1}}\ [D]_{t_{q+1}t_{q+2}}
\end{aligned} \tag{3.7}$$

where, if we multiply two different types together, then their indices must agree at the position of multiplication. Moreover, the first and the last indices of a product of matrices of these three types must be the same. Then these operators generate a subalgebra $\mathscr{V}$ of differential operators in $\mathscr{U}$ that commute with the operators $L_{\alpha_{p_i}\beta_{p_i}}$, with $\alpha_{p_i}<\beta_{p_i}$, $1\leqslant i\leqslant r$, where $1\leqslant u_i,\ v_s,\ t_j\leqslant r$, $1\leqslant i, j, s\leqslant q$ and q is a non-negative integer.

Proof. Let Γ' denote the complexification of $\mathrm{so}^*(2n)$, that is $\Gamma'=\mathrm{so}(2n, C)$, the Lie algebra of the complex $2n\times 2n$ special orthogonal group. If $\xi\in\Gamma'$, then ξ is of the following form:

$$\xi=\begin{pmatrix} Y & X\\ W & Q\end{pmatrix}$$

where $[Y]$ is a $n\times n$ complex matrix, $[Q]=-[Y]^T$ and $[W]$, $[X]$ are $n\times n$ skew-symmetric matrices (the Lie algebra of $\mathrm{SO}(2n, C)$ can be written in several ways. The form that we are using here is the most convenient for our purpose). On the other hand, the differential operators $P_{\alpha\beta}$, $D_{\alpha\beta}$ and $L_{\alpha\beta}$ as defined in (3.2) also form a basis for Γ'. Let $S(\mathrm{so}(2n, C))$ denote the symmetric algebra of $\mathrm{so}(2n, C)$ and $\mathscr{U}(\mathrm{so}(2n, C))$ denote the universal enveloping algebra of $\mathrm{so}(2n, C)$. We can now define the coadjoint representation T of $H'=\mathrm{SO}(2n, C)$ in $S(\mathrm{so}(2n, C))$ by the equation

$$[T(h')p](\xi)=p(h'^{-1}\xi h') \qquad h'\in H' \qquad p\in S(\mathrm{so}(2n, C)) \qquad \text{and } \xi\in\Gamma'.$$

A polynomial $p\in S(\mathrm{so}(2n, C))$ is said to be K-invariant, where K is a subgroup of H', if $T(k)p=p$, for all $k\in \mathrm{K}$.

We now have the canonical isomorphism ϕ of $S(\mathrm{so}(2n, C))$ onto $\mathscr{U}(\mathrm{so}(2n, C))$ defined as follows (cf. [14]):

Suppose $p\in S(\mathrm{so}(2n, C))$, then p can be expressed uniquely as

$$p(\xi)=\sum_{s\leqslant d} a_{i_1j_1\ldots i_sj_s} y_{i_1j_1}\ldots x_{i_aj_a}\ldots w_{i_sj_s}$$

where the coefficients $a_{i_1j_1\ldots i_sj_s}$ are symmetric functions, that is, $a_{i_{\sigma(1)}j_{\sigma(1)}\ldots i_{\sigma(s)}j_{\sigma(s)}}=a_{i_1j_1\ldots i_sj_s}$, for all permutations σ in the symmetric group of order s and for all integers s less than or equal to a fixed integer d. Now, $\phi: S(\mathrm{so}(2n, C))\to\mathscr{U}(\mathrm{so}(2n, C))$ is defined by

$$\phi(p)=\sum_{s\leqslant d} a_{i_1j_1\ldots i_sj_s} L_{i_1j_1}\ldots P_{i_aj_a}\ldots D_{i_sj_s}.$$

Let K′ denote the subgroup $GL(p_1, C)\times\ldots\times GL(p_r, C)$ of $GL(n, C)$. Then K′ is embedded in $GL(n, C)$ as a block diagonal subgroup:

$$\begin{pmatrix} k_1' & & 0 \\ & \ddots & \\ 0 & & k_r' \end{pmatrix} \qquad k_i' \in GL(p_i, C) \qquad 1\leqslant i\leqslant r.$$

An element $u\in\mathscr{U}(so(2n, \mathbf{C}))$ is said to be K′-invariant if the conditions

$$[u, L_{\alpha_{p_i}\beta_{p_i}}]=0 \qquad \text{for all } L_{\alpha_{p_i}\beta_{p_i}} \qquad 1\leqslant i\leqslant r$$

are satisfied. Observe that $L_{\alpha_{p_i}\beta_{p_i}}$ generate the Lie algebra of K′.

It is well known that the map ϕ carries the K′-invariant polynomials onto the K′-invariant differential operators (cf [14]). Thus, to show that a differential operator of the form 3.7 is K′-invariant, it suffices to show that its inverse image under the map ϕ is a K′-invariant polynomial function. For this, let us partition Y, X, W and Q in the same way as the matrix P. Now, let

$$k'=\begin{pmatrix} k_1 & & & & & \\ & \ddots & & & 0 & \\ & & k_r & & & \\ & & & \tilde{k}_1 & & \\ & 0 & & & \ddots & \\ & & & & & \tilde{k}_r \end{pmatrix}\in SO(2n, C)$$

where $k_i, \tilde{k}_i\in GL(p_i, C), \tilde{k}_i=(k_i^T)^{-1}, 1\leqslant i\leqslant r$.

Let $\xi\in\Gamma'$, then the matrix $k'^{-1}\xi k'$ can be written as

$$k'^{-1}\xi k'=\begin{pmatrix} k_1^{-1}[Y]_{11}k_1 & \cdots & k_1^{-1}[Y]_{1r}k_r & k_1^{-1}[X]_{11}\tilde{k}_1 & \cdots & k_1^{-1}[X]_{1r}\tilde{k}_r \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ k_r^{-1}[Y]_{r1}k_1 & \cdots & k_r^{-1}[Y]_{rr}k_r & k_r^{-1}[X]_{r1}\tilde{k}_1 & \cdots & k_r^{-1}[X]_{rr}\tilde{k}_r \\ \tilde{k}_1^{-1}[W]_{11}k_1 & \cdots & \tilde{k}_1^{-1}[W]_{1r}k_r & \tilde{k}_1^{-1}[Q]_{11}\tilde{k}_1 & \cdots & \tilde{k}_1^{-1}[Q]_{1r}\tilde{k}_r \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \tilde{k}_r^{-1}[W]_{r1}k_1 & \cdots & \tilde{k}_r^{-1}[W]_{rr}k_r & \tilde{k}_r^{-1}[Q]_{r1}\tilde{k}_1 & \cdots & \tilde{k}_r^{-1}[Q]_{rr}\tilde{k}_r \end{pmatrix}.$$

If f_1 is the inverse image under the map ϕ of the type 1 in (3.7), then f_1 is given by

$$f_1(\xi)=\mathrm{Tr}([Y]_{u_1u_2}[Y]_{u_2u_3} \quad \ldots \quad [Y]_{u_qu_{q+1}}).$$

If $u_{q+1}=u_1$, then

$$\begin{aligned}[][T(k')f_1](\xi)&=f_1(k'^{-1}\xi k')\\ &=\mathrm{Tr}((k_{u_1}^{-1}[Y]_{u_1u_2}k_{u_2})(k_{u_2}^{-1}[Y]_{u_2u_3}k_{u_3}) \quad \ldots \quad (k_{u_q}^{-1}[Y]_{u_qu_1}k_{u_1}))\\ &=\mathrm{Tr}(k_{u_1}^{-1}[Y]_{u_1u_2}\ldots[Y]_{u_qu_1}k_{u_1})\\ &=\mathrm{Tr}([Y]_{u_1u_2}\ldots[Y]_{u_qu_1})\\ &=f_1(\xi).\end{aligned}$$

If f_2 is the inverse image of ϕ type 2 in (3.7), then f_2 is given by

$$f_2(\xi)=\mathrm{Tr}([X]_{v_1v_2}[W]_{v_2v_3}\dots[X]_{v_qv_{q+1}}[W]_{v_{q+1}v_{q+2}}).$$

Suppose $v_{q+2}=v_1$. Now

$$\begin{aligned}[T(k')f_2](\xi)&=f_2(k'^{-1}\xi k')\\ &=\mathrm{Tr}((k_{v_1}^{-1}[X]_{v_1v_2}\bar{k}_{v_2})(\bar{k}_{v_2}^{-1}[W]_{v_2v_3}k_{v_3})\dots(k_{v_q}^{-1}[X]_{v_qv_{q+1}}\bar{k}_{v_{q+1}})(\bar{k}_{v_{q+1}}^{-1}[W]_{v_{q+1}v_1}k_{v_1}))\\ &=\mathrm{Tr}(k_{v_1}^{-1}[X]_{v_1v_2}[W]_{v_2v_3}\dots[X]_{v_qv_{q+1}}[W]_{v_{q+1}v_1}k_{v_1})\\ &=\mathrm{Tr}([X]_{v_1v_2}[W]_{v_2v_2}\dots[X]_{v_qv_{q+1}}[W]_{v_{q+1}v_1})\\ &=f_2(\xi).\end{aligned}$$

If f_3 is the inverse image of ϕ of type 3 in (3.7), then f_3 is given by

$$f_3(\xi)=\mathrm{Tr}([X]_{t_1t_2}[Q]_{t_2t_3}\dots[Q]_{t_qt_{q+1}}[W]_{t_{q+1}t_{q+2}}).$$

Suppose $t_{q+2}=t_1$, then

$$\begin{aligned}[T(k')f_3](\xi)&=f_3(k'^{-1}\xi k')\\ &=\mathrm{Tr}((k_{t_1}^{-1}[X]_{t_1t_2}\bar{k}_{t_2})(\bar{k}_{t_2}^{-1}[Q]_{t_2t_3}\bar{k}_{t_3})\dots(\bar{k}_{t_q}^{-1}[Q]_{t_qt_{q+1}}\bar{k}_{t_{q+1}})(\bar{k}_{t_{q+1}}^{-1}[W]_{t_{q+1}t_1}k_{t_1}))\\ &=\mathrm{Tr}(k_{t_1}^{-1}[X]_{t_1t_2}[Q]_{t_2t_3}\dots[Q]_{t_qt_{q+1}}[W]_{t_{q+1}t_1}k_{t_1})\\ &=\mathrm{Tr}([X]_{t_1t_2}[Q]_{t_2t_3}\dots[Q]_{t_qt_{q+1}}[W]_{t_{q+1}t_1})\\ &=f_3(\xi).\end{aligned}$$

Now, it is clear that besides the commuting operators obtained from type 1, 2 and 3, we can multiply them together, as long as the products satisfy the conditions on the indices as stated in the theorem. Up to now, we are dealing with the algebra $\mathscr{U}(\mathrm{so}(2n, C))$. However, it is easy to see that the commuting operators we have obtained also lie in the universal enveloping algebra $\mathscr{U}$. Since any commuting operator in $\mathscr{U}$ also belongs to $\mathscr{U}(\mathrm{so}(2n, C))$, we have found all generators of the commuting operators we desired in $\mathscr{U}$. Therefore, the proof of the theorem is now completed. Q.E.D.

Remark 3.8. In [5], Klink and Ton-That obtained a subalgebra of differential operators generated by the operators of type (1) in theorem 3.6. Our result includes their operators because U(n) is a subgroup of SO*(2n), as mentioned before. In addition, because of the three matrices P, D, and E, we have established two new types of commuting differential operators in $\mathscr{U}$, namely, type (2) and (3) in theorem 3.6. This gives us more choices of operators for decomposing our tensor products. It is in this sense that we generalized the result in [5].

Example. We have the following commuting operator:

$$\mathrm{Tr}([L]_{u_1u_2}[L]_{u_2u_3}\quad\dots\quad[L]_{u_qv_1}\quad[P]_{v_1t_2}\quad[E]_{t_2t_3}\quad\dots\quad[E]_{t_st_{s+1}}\quad[D]_{t_{s+1}u_1}).$$

For $1\leqslant i\leqslant r$, we define the matrix

$$R^{(p_i)*}=\begin{pmatrix}0 & I_k\\ -I_k & 0\end{pmatrix}(R^{(p_i)})^T\begin{pmatrix}0 & -I_k\\ I_k & 0\end{pmatrix}$$

where $(R^{(p_i)})^T$ is the transpose of $R^{(p_i)}$. Then we have the following:

Theorem 3.9. The differential operators of the form

$$\mathrm{Tr}(A^{(i_1)}\ldots A^{(i_j)}\ldots A^{(i_s)}) \qquad A^{(i_j)}=R^{(p_{i_j})} \text{ or } R^{(p_{i_j})^*} \tag{3.10}$$

for $1\leqslant i_j\leqslant r$, where s is a non-negative integer, generate the same subalgebra $\mathscr{V}$ of invariant operators as the differential operators defined in theorem 3.4 (the indices $i_1,\ldots,i_s$ may not be distinct).

Proof. The proof of this theorem is exactly the same as the proof of lemma 6 in [15]. Q.E.D.

At this point, we have found all operators in $\mathscr{U}$ which commute with the operators $L_{\alpha_{p_i}\beta_{p_i}}$, with $\alpha_{p_i}<\beta_{p_i}$, $1\leqslant i\leqslant r$. Therefore, we only need to find operators in $\mathscr{V}$ that commute with the operators $D_{\alpha_{p_i}\beta_{p_i}}$ with $\alpha_{p_i}<\beta_{p_i}$, $1\leqslant i\leqslant r$, and which decompose $\mathrm{Ker}^{(m)(M)}_{\max}$ into distinct one-dimensional eigenspaces. The way of constructing commuting operators as shown in the proof of 3.6 is not suitable in this case because the operators $D_{\alpha_{p_i}\beta_{p_i}}$, do not embed in the Lie algebra $\mathrm{SO}^*(2n)$ diagonally as $L_{\alpha_{p_i}\beta_{p_i}}$,. Here, we shall construct our commuting operators by 'brute force' with the help of theorem 3.9. The explicit form of certain of such commuting operators are given in the following:

Theorem 3.11. In the algebra $\mathscr{V}$, the differential operators of the form

$$\mathrm{Tr}\left(\sum(-1)^q A^{(i_1)}\ldots A^{(i_j)}\ldots A^{(i_s)}\right) \qquad A^{(i_j)}=R^{(p_{i_j})} \text{ or } R^{(p_{i_j})*}$$

where $i_1,\ldots,i_s$ are fixed indices and may not be distinct, s is a non-negative integer, q is the number of $A^{(i_j)}$ that represents $R^{(p_{i_j})^*}$ and the sum is over all possible variation for each $R^{(p_{i_j})^*}$ and $R^{(p_{i_j})}$, generate a subalgebra $\mathscr{Y}$ of differential operators in $\mathscr{V}$ that commute with the differential operators $D_{\alpha_{p_i}\beta_{p_i}}$, with $\alpha_{p_i}<\beta_{p_i}$, $1\leqslant i\leqslant r$.

Proof. To avoid cumbersome notation, we use the Einstein convention. It was shown in [16] that the operators

$$R'^{(p_i)}_{\alpha\beta}=R^{(p_i)}_{\alpha\beta}-R^{(p_i)}_{\beta+k\,\alpha+k}=-R'^{(p_i)}_{\beta+k\,\alpha+k} \qquad R'^{(p_i)}_{\alpha\,\beta+k}=R^{(p_i)}_{\alpha\,\beta+k}+R^{(p_i)}_{\beta\,\alpha+k}$$
$$R'^{(p_i)}_{\alpha+k\,\beta}=R^{(p_i)}_{\alpha+k\,\beta}+R^{(p_i)}_{\beta+k\,\alpha} \qquad 1\leqslant\alpha \qquad \beta\leqslant k$$

commute with $D_{\alpha_{p_i}\beta_{p_i}}$, with $\alpha_{p_i}<\beta_{p_i}$, $1\leqslant i\leqslant r$.

Now, let $R^{(p_i)*}_{\alpha\beta}$ denote the $\alpha\ \beta$ entry of the matrix $R^{(p_i)*}$, then we have

$$R^{(p_i)*}_{\alpha\beta}=R^{(p_i)}_{\beta+k\,\alpha+k} \qquad R^{(p_i)*}_{\alpha+k\,\beta+k}=R^{(p_i)}_{\beta\alpha} \qquad R^{(p_i)*}_{\alpha\,\beta+k}=-R^{(p_i)}_{\beta\,\alpha+k}$$

and

$$R^{(p_i)*}_{\alpha+k\,\beta}=-R^{(p_i)}_{\beta+k\,\alpha} \qquad 1\leqslant\alpha \qquad \beta\leqslant k.$$

Now, any term in $\mathrm{Tr}(\Sigma(-1)^q A^{(i_1)}\ldots A^{(i_s)})$ is given by

$$\tilde{R}^{(i_1)}_{\alpha_1\alpha_2}\tilde{R}^{(i_2)}{\alpha_2\alpha_3}\ldots\tilde{R}^{(i_j)}_{\alpha_j\alpha_{j+1}}\ldots\tilde{R}^{(i_s)}_{\alpha_s\alpha_1} \qquad \text{where } \tilde{R}^{(i_j)}_{\alpha_j\alpha_{j+1}}=R^{(p_{i_j})}_{\alpha_j\alpha_{j+1}} \text{ or } R^{(p_{i_j})*}_{\alpha_j\alpha_{j+1}}.$$

If we consider the product $R'^{(p_{i_1})}_{\alpha_1\alpha_2}R'^{(p_{i_2})}_{\alpha_2\alpha_3}\ldots R'^{(p_{i_s})}_{\alpha_s\alpha_1}$, for all possible indices $\alpha_1,\alpha_2,\ldots,\alpha_s$, then the sum

$$\sum R'^{(p_{i_1})}_{\alpha_1\alpha_2}R'^{(p_{i_2})}_{\alpha_2\alpha_3}\ldots R'^{(p_{i_s})}_{\alpha_s\alpha_1}$$

(over all possible indices) is equal to $\mathrm{Tr}(\Sigma(A^{(i_1)}\ldots A^{(i_s)}))$, up to signs, where the sum is over all possible variation for each $R^{(p_{i_j})*}$ and $R^{(p_{i_j})}$. Now, the signs agree if we change an even number of $R^{(p_{i_j})}$ to $R^{(p_{i_j})*}$ and the signs are different if we change an odd number of $R^{(p_{i_j})}$ to $R^{(p_{i_j})*}$. As a result

$$\sum R'^{(p_{i_1})}_{\alpha_1\alpha_2} R'^{(p_{i_2})}_{\alpha_2\alpha_3}\ldots R'^{(p_{i_s})}_{\alpha_s\alpha_1} = \mathrm{Tr}\left(\sum(-1)^q A^{(i_1)}\ldots A^{(i_s)}\right)$$

where q is the number of $A^{(i_j)}$ that represents $R^{(p_{i_j})*}$. Clearly,

$$\sum R'^{(p_{i_1})}_{\alpha_1\alpha_2} R'^{(p_{i_2})}_{\alpha_2\alpha_3}\ldots R'^{(p_{i_s})}_{\alpha_s\alpha_1}$$

commutes with $D_{\alpha_{p_i}\beta_{p_i}}$, with $\alpha_{p_i}<\beta_{p_i}$, $1\leqslant i\leqslant r$. Therefore, $\mathrm{Tr}(\Sigma(-1)^q A^{(i_1)}\ldots A^{(i_s)})$ commutes with the differential operators $D_{\alpha_{p_i}\beta_{p_i}}$, with $\alpha_{p_i}<\beta_{p_i}$, $1\leqslant i\leqslant r$, and $\mathrm{Tr}(\Sigma(-1)^q A^{(i_1)}\ldots A^{(i_s)})$ is obviously an element of $\mathscr{V}$. This complete the proof of our theorem. Q.E.D.

Example. We have the following commuting operator in $\mathscr{Y}$:

$$\mathrm{Tr}(R^{(p_1)}R^{(p_2)}+R^{(p_1)*}R^{(p_2)*}-R^{(p_1)*}R^{(p_2)}-R^{(p_1)}R^{(p_2)*}).$$

To be useful for our programme of multiplicity breaking, the commuting operators must be Hermitian. Therefore, we want to know what the adjoint of an operator in $\mathscr{Y}$ looks like. Instead of examining the adjoints of the operators defined in theorem 3.6 directly, it is more convenient to check the adjoints of the commuting operators using the forms defined in theorem 3.9. First we have the following:

Proposition 3.12. The adjoint of

$$\mathrm{Tr}(A^{(i_1)}\ldots A^{(i_j)}\ldots A^{(i_s)})$$

where $A^{(i_j)}=R^{(p_{i_j})}$ or $\mathrm{R}^{(p_{i_j})*}$, is

$$\mathrm{Tr}(A^{(i_s)}\ldots A^{(i_1)})$$

therefore, the adjoint of an element in $\mathscr{V}$ is still in $\mathscr{V}$.

Proof. It was shown in [5] that the adjoint of

$$R_{ij}=z_{\eta i}\frac{\partial}{\partial z_{\eta j}}$$

where η ranges over an appropriate set of indices, is equal to

$$R_{ji}=z_{\eta j}\frac{\partial}{\partial z_{\eta i}}.$$

Here, we also use the Einstein convention. Once again, we let $R^{(i_j)*}_{\alpha\beta}$ denote the $\alpha\ \beta$ entry of the matrix $R^{(p_{i_j})*}$. Then we have

$$\mathrm{Tr}(A^{(i_1)}\ldots A^{(i_s)})=\tilde{R}^{(i_1)}_{\alpha_1\alpha_2}\tilde{R}^{(i_2)}_{\alpha_2\alpha_3}\ldots\tilde{R}^{(i_s)}_{\alpha_s\alpha_1}$$

where $\tilde{R}^{(i_j)}_{\alpha_j\alpha_{j+1}}=R^{(i_j)}_{\alpha_j\alpha_{j+1}}$ if $A^{(i_j)}=R^{(p_{i_j})}$ or $\tilde{R}^{(i_j)}_{\alpha_j\alpha_{j+1}}=R^{(i_j)*}_{\alpha_j\alpha_{j+1}}$ if $A^{(i_j)}=R^{(p_{i_j})}$.

Now, if $\mathcal{O}$ is an operator, then let $\mathcal{O}^*$ denote the adjoint of $\mathcal{O}$. It follows that

$$\begin{aligned}(\mathrm{Tr}(A^{(i_1)}\dots A^{(i_s)})^* &= (\tilde{R}^{(i_1)}_{\alpha_1\alpha_2}\tilde{R}^{(i_2)}_{\alpha_2\alpha_3}\dots\tilde{R}^{(i_s)}_{\alpha_s\alpha_1})^* \\ &= (\tilde{R}^{(i_s)}_{\alpha_s\alpha_1})^*(\tilde{R}^{(i_1)}_{\alpha_1\alpha_2}\tilde{R}^{(i_2)}_{\alpha_2\alpha_3}\dots\tilde{R}^{(i_{s-1})}_{\alpha_{s-1}\alpha_s})^* \\ &= \tilde{R}^{(i_s)}_{\alpha_1\alpha_s}\tilde{R}^{(i_{s-1})}_{\alpha_s\alpha_{s-1}}\dots\tilde{R}^{(i_1)}_{\alpha_{s-1}\alpha_1} \\ &= \mathrm{Tr}(A^{(i_s)}\dots A^{(i_1)}).\end{aligned}$$

Hence, the proof is completed. Q.E.D.

Now, we can give the explicit form of the adjoint of an operator in $\mathscr{Y}$ in the following:

Proposition 3.13. The adjoint of

$$\mathrm{Tr}\Big(\sum(-1)^q A^{(i_1)}\dots A^{(i_s)}\Big)$$

is

$$\mathrm{Tr}\Big(\sum(-1)^q A^{(i_s)}\dots A^{(i_1)}\Big).$$

Hence, the adjoint of an element in $\mathscr{Y}$ is still in $\mathscr{Y}$.

Proof. It follows from the fact that the summation is over all possible variation of $R^{(p_{i_j})*}$ and $R^{(p_{i_j})}$, $1\leqslant j\leqslant s$, and proposition 3.12 Q.E.D.

Remark 3.14. It is well known that the sum of an operator and its adjoint is Hermitian. Therefore, if an operator $\mathcal{O}$ in $\mathscr{Y}$ is not Hermitian, then since its adjoint $\mathcal{O}^*$, is still in $\mathscr{Y}$ according to the previous result, we can use the Hermitian operator $\mathcal{O}+\mathcal{O}^*$. As a result, we can always find a Hermitian operator in $\mathscr{Y}$ and use it to decompose the $\mathrm{Ker}^{(m)(M)}_{\max}$ space into distinct one-dimensional subspaces. In general we choose the commuting differential operator in an *ad hoc* manner. However, in practice, just a low degree differential operator generally suffices. Moreover, it has been shown in [17] that all the differential operators defined in theorem 3.9 can be generated from a finite set. In the next section, we shall illustrate the procedure by an example.

4. An example

In this section, let us consider the decomposition of the two-fold tensor product $V^{(2,1)}_{\mathrm{Sp}}\otimes V^{(2,0)}_{\mathrm{Sp}}$ of Sp(4, $\boldsymbol{C}$). According to our program of multiplicity breaking, we consider the Fock space $\mathscr{F}(\boldsymbol{C}^{3\times 4})$, which contains the Sp(4, $\boldsymbol{C}$)-module $P^{(2,1,2)}(\boldsymbol{C}^{3\times 4})$ with $p_1=2$ and $p_2=1$. The module $P^{(2,1,2)}(\boldsymbol{C}^{3\times 4})$ in turn contains the submodule $H^{(2,1,2)}(\boldsymbol{C}^{3\times 4})$. By theorem 2.5, $H^{(2,1,2)}(\boldsymbol{C}^{3\times 4})$ is isomorphic to $V^{(2,1)}_{\mathrm{Sp}}\otimes V^{(2,0)}_{\mathrm{Sp}}$ and by theorem 2.1, $P^{(2,1,2)}(\boldsymbol{C}^{3\times 4})$ is isomorphic to $V^{(2,0)}_{\mathrm{Sp}}\otimes V^{(1,0)}_{\mathrm{Sp}}\otimes V^{(2,0)}_{\mathrm{Sp}}$.

Now, using the formula in remark 2.2, we have the following two decompositions:

$$(2,1)\otimes(2,0)=(4,1)\oplus(3,2)\oplus(3,0)\oplus 2(2,1)\oplus(1,0)$$

and

$$(2,0)\otimes(1,0)\otimes(2,0)=(5,0)\oplus 2(4,1)\oplus 2(3,2)\oplus 3(3,0)\oplus 4(2,1)\oplus 3(1,0).$$

Every irreducible representation in the decomposition of $V_{\mathrm{Sp}}^{(2,1)}\otimes V_{\mathrm{Sp}}^{(2,0)}$ has multiplicity one except $V_{\mathrm{Sp}}^{(2,1)}$. Therefore, we only need to deal with the irreducible module $V_{\mathrm{Sp}}^{(2,1)}$ with multiplicity two in $H^{(2,1,2)}(C^{3\times 4})$. Recall that $H^{(2,1,2)}(C^{3\times 4})$ consists of all polynomial functions in $P^{(2,1,2)}(C^{3\times 4})$ which are simultaneously annihilated by the operators

$$L_{12}=\sum_{\eta=1}^{4} Z_{1\eta}\frac{\partial}{\partial Z_{2\eta}} \qquad \text{and} \qquad D_{12}=\sum_{\eta=1}^{2}\left(\frac{\partial^2}{\partial Z_{1\eta+2}\,\partial Z_{2\eta}}-\frac{\partial^2}{\partial Z_{1\eta}\,\partial Z_{2\eta+2}}\right).$$

If f_0 is the highest weight vector of $V_{\mathrm{Sp}}^{(2,1)}$, then it is given by $f_0=z_{11}(z_{11}z_{22}-z_{12}z_{21})$.

We use the dual pair (SO*(6), Sp(4, C)). The multiplicity of $V_{\mathrm{Sp}}^{(2,1)}$ in the Sp(4, C)-module $P^{(2,1,2)}(C^{3\times 4})$ is four and there exist four linearly independent differential operators

$$L_{32}P_{23} \qquad L_{31}P_{13} \qquad P_{12}L_{31}L_{32} \qquad \text{and} \qquad D_{12}P_{23}P_{13}$$

which form a basis for $\mathrm{Hom}_G(V_{\mathrm{Sp}}^{(2,1)}, P^{(2,1,2)}(C^{3\times 4}))$, where

$$L_{ij}=\sum_{\eta=1}^{4} Z_{i\eta}\frac{\partial}{\partial Z_{j\eta}} \qquad P_{ij}=\sum_{\eta=1}^{2}(Z_{i\eta+2}Z_{j\eta}-Z_{i\eta}Z_{j\eta+2})$$

and

$$D_{ij}=\sum_{\eta=1}^{2}\left(\frac{\partial^2}{\partial Z_{i\eta+2}\,\partial Z_{j\eta}}-\frac{\partial^2}{\partial Z_{i\eta}\,\partial Z_{j\eta+2}}\right) \qquad 1\leqslant i \qquad j\leqslant 3.$$

At this point, we want to mention how to choose the above four linearly independent operators. Our goal is to find four elements in $\mathscr{U}$(SO*(6)) that send $V_{\mathrm{Sp}}^{(2,1)}$ into $P^{(2,1,2)}(C^{3\times 4})$. In $\mathscr{U}$(SO*(6)), the raising operators are $P_{\alpha\beta}$ and $L_{\alpha\beta}$ for $\alpha>\beta$ and the lowering operators are $P_{\alpha\beta}$ and $D_{\alpha\beta}$ for $\alpha<\beta$. Therefore, we want to combine certain raising and lowering operators in $\mathscr{U}$(SO*(6)) so that we can raise the 2-tuple of integers (2, 1) to (2, 1, 2).

Let us return to our example. The space $W_{\max}^{(2,1)(2,1,2)}$ is spanned by

$$f_1=L_{32}P_{23}f_0 \qquad f_2=L_{31}P_{13}f_0 \qquad f_3=P_{12}L_{31}L_{32}f_0 \qquad \text{and}\ f_4=D_{12}P_{23}P_{13}f_0.$$

Now, let $\mathrm{Ker}_{\max}^{(2,1)(2,1,2)}$ denote the projection of $W_{\max}^{(2,1)(2,1,2)}$ into $H^{(2,1,2)}(C^{3\times 4})$, that is, $\mathrm{Ker}_{\max}^{(2,1)(2,1,2)}$ is the common kernel space of the operators L_{12} and D_{12}. Then the application of L_{12} and D_{12} to a general vector in $W_{\max}^{(2,1)(2,1,2)}$ of the form $\alpha_1 f_1+\alpha_2 f_2+\alpha_3 f_3+\alpha_4 f_4$, $\alpha_i\in C$, leads to a two-dimensional subspace which is spanned by the vectors $3/2f_1+3/2f_2+f_3$ and $2f_2+f_4$. Hence, $\mathrm{Ker}_{\max}^{(2,1)(2,1,2)}$ is a two-dimensional subspace. We now use the Casimir operator

$$C_0\equiv-(1/2)\,\mathrm{Tr}(R^{(1)}R^{(2)}-R^{(1)}R^{(2)*}-R^{(1)*}R^{(2)}+R^{(1)*}R^{(2)*})$$

where

$$R^{(1)}=(R_{ij}^{(1)})\ \text{with}\ R_{ij}^{(1)}=\sum_{\eta=1}^{2} Z_{\eta_i}\frac{\partial}{\partial Z_{\eta_j}}$$

and

$$R^{(2)}=(R_{ij}^{(2)}) \qquad \text{with } R_{ij}^{(2)}=Z_{3i}\frac{\partial}{\partial Z_{3j}} \qquad 1\leqslant i \qquad j\leqslant 4.$$

It is easy to see that $R_{\alpha\beta}^{(1)}$ and $R_{\gamma\tau}^{(2)}$ commutes with each other since they involve different set of variables. Therefore, the Casimir operator C_0 is Hermitian. The Casimir operator C_0 acting on $\mathrm{Ker}_{\max}^{(2,1)(2,1,2)}$ has two distinct eigenvalues $\lambda_1=6$ and $\lambda_2=1$. The coresponding eigenvector for λ_1 is $h_1=2f_2+f_4$ and the corresponding eigenvector for λ_2 is $h_2=3/2f_1-1/2f_2+f_3-3/2f_4$. h_1 and h_2 are orthogonal vectors since $\lambda_1\neq\lambda_2$ and C_0 is Hermitian. In conclusion, the two intertwining operators that send $V_{\mathrm{Sp}}^{(2,1)}$ into two orthogonal (equivalent) submodules of $H^{(2,1,2)}(C^{3\times 4})$ are

$$P_1=2L_{31}P_{13}+D_{12}P_{23}P_{13}$$

and

$$P_2=3/2L_{32}P_{23}-1/2L_{31}P_{13}+P_{12}L_{31}L_{32}-3/2D_{12}P_{23}P_{13}$$

and any element $f\in V_{\mathrm{Sp}}^{(2,1)}$ can be mapped with these operators into $H^{(2,1,2)}(C^{3\times 4})$.

5. Conclusion

We have shown how to decompose an r-fold tensor product of arbitrary irreducible representations of $\mathrm{Sp}(2k, C)$, by finding generalized Casimir operators whose eigenvalues and eigenvectors can be used to resolve the ambiguity occuring when equivalent representations appear more than once in the decomposition. We first embed the tensor product space $H^{(M)}$ into a bigger polynomial space. Then, maps that carry an irreducible representation space, labelled by (m), into the tensor product space are constructed.

To obtain an orthogonal direct sum copies of (m) in $H^{(M)}$, we have constructed an algebra of operators which leaves $H^{(M)}$ invariant. Certain elements of these commuting operators are Hermitian. Therefore, their eigenvalues can be used to form an orthogonal direct sum of the copies of (m) in $I(V_{\mathrm{Sp}}^{(m)})\cap H^{(M)}$.

It is possible to implement such a procedure on a computer and our immediate goal is to write a computer program for the procedure. In fact, some of the calculations in section 4 were computed using the computer program Mathematica. This procedure of multiplicity breaking of tensor products can be applied to other Lie groups. We intend to investigate such a problem in a further publication.

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